Anisotropic diffusion of the state of polarization in optical fibers with randomly varying birefringence

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Polarization diffusion in communication fibers is studied. Diffusion of the states of polarization in an optical fiber is found to be anisotropic on the surface of the Poincaré sphere. The predicted anisotropy has significant implications for nonlinear evolution in long-distance communication systems. © 1995 Optical Society of America

Random fluctuations in birefringence along an optical fiber degrade the transmission rate in communication systems. In modern, dispersion-shifted communication fibers, at typical data rates and transmission powers, the birefringent beat length is of the order of meters or tens of meters, whereas the dispersive and nonlinear scale lengths are typically hundreds of kilometers. Thus, although the index difference corresponding to the birefringence is small, \( \Delta n \approx 10^{-7} \), the birefringence should be considered large, and its effect would be devastating except that the orientation of this birefringence is rapidly and randomly changing on a length scale of tens to hundreds of meters. Under these circumstances, if one assumes that the variation of the birefringence is so rapid that its only effect is to scramble the electric field on the Poincaré sphere, one finds that light propagation in both nonreturn-to-zero and soliton communication systems can be described by the Manakov equation. If one takes into account additional terms that act as noise sources are added to the Manakov equation. These additional noise terms are both linear and nonlinear in field strength. The linear terms lead to the usual linear polarization mode dispersion, whereas the nonlinear terms will lead to a nonlinear polarization mode dispersion whose effects have barely begun to be explored.

In simulations of long-distance propagation in optical fibers, the practice has often been simply to scramble the electric field on the Poincaré sphere at fixed intervals while allowing the field to evolve deterministically in between. This approach tacitly assumes that the electric field diffuses uniformly on the Poincaré sphere. However, communication fibers are nearly linearly birefringent so that the polarization eigenaxes, even though randomly varying, are confined to the equator of the Poincaré sphere, and one might anticipate that the equatorial diffusion rate on the Poincaré sphere is not the same as the azimuthal diffusion rate. In this Letter we show that the electric-field diffusion on the Poincaré sphere is indeed anisotropic. This anisotropy affects the relative contribution of the linear and the nonlinear noise terms and thus has important implications for nonlinear pulse evolution.

In this Letter we use two different physical models to study the rate of diffusion of an input state of polarization on the Poincaré sphere as a function of fiber autocorrelation length \( h_{\text{fiber}} \) and beat length \( L_B \). We may measure the diffusion lengths by calculating the variances of the Stokes parameters \( S_1, S_2, \) and \( S_3 \) and determining the lengths \( d_1, d_2, \) and \( d_3 \) at which the variances are within \( 1/e \) of the final asymptotic value of 1/3. The diffusion lengths \( d_1 \) and \( d_2 \) correspond to equatorial diffusion along the Poincaré sphere. Both these lengths and the polarization decorrelation length \( h_p \) are physically related to the length over which the electric field loses memory of its orientation. By contrast, \( d_3 \) corresponds to azimuthal diffusion along the Poincaré sphere and is physically related to the length over which the electric field loses memory of the ratio between the minor and major axes of the polarization ellipse.

After we remove variation common to both polarizations components, the spatial dependence of the electric field \( \mathbf{U}(r, \omega, z) \) is given by

\[
\frac{\partial \mathbf{U}(r, \omega, z)}{\partial z} = \begin{bmatrix} x & y \\ -x & y \end{bmatrix} \mathbf{U}(r, \omega, z),
\]

where \((x^2 + y^2)^{1/2}\) is the birefringence. The beat length is given by \( L_B = 2\pi/(x^2 + y^2)^{1/2} \). The coupling constant \( y \) is taken to be real because the fiber is assumed to be linearly birefringent, as is nearly the case for real communication fibers.

In the first model we assume the strength of the birefringence to be fixed but we allow the orientation to vary randomly, i.e., \( x(\omega, z) = b(\omega) \cos \theta(z) \) and \( y(\omega, z) = b(\omega) \sin \theta(z) \), where \( b(\omega) \) does not depend on the distance \( z \). We further assume that the rate of change of the angle \( \theta \) of the orientation axes is a white-noise process, i.e., \( \partial \theta/\partial z = g_\theta(z), \) \( \langle g_\theta(z) \rangle = 0, \) and \( \langle g_\theta(z)g_\theta(z + u) \rangle = \Gamma \delta(u) \), where \( \Gamma \) is a constant and \( \delta(u) \) is the Dirac delta function. The fiber autocorrelation length \( h_{\text{fiber}} \) is \( 2\Gamma \). In the second model, both \( x \) and \( y \) vary independently according to the following Langevin equations:

\[
dx/dz = -\alpha x + g_\omega(\omega, z),
\]

and

\[
dy/dz = -\alpha y + g_\omega(\omega, z),
\]

where \( \alpha \) is a constant and both \( g_\omega(\omega, z) \) and \( g_\omega(\omega, z) \) are white-noise processes with zero mean and the same distribution. The fiber autocorrelation length \( h_{\text{fiber}} \) is \( 1/\alpha \).

We now switch to the Poincaré representation of the field. Instead of following the evolution of the complex
fields \( U_1(z) \) and \( U_2(z) \), we will follow the evolution of the three real Stokes parameters\(^1\) \( S_1 = U_1^* U_1^* - U_2 U_2^* \), \( S_2 = U_1^* U_2 + U_2 U_1^* \), and \( S_3 = -i(U_1^* U_2 - U_2 U_1^*) \), where \( U_1 = U \cdot \mathbf{e}_1(z) \) and \( U_2 = U \cdot \mathbf{e}_2(z) \). We assume that the field \( U \) is normalized so that \( S_1^2 + S_2^2 + S_3^2 = 1 \). There are two physically sensible choices of the orthogonal unit vectors \( \mathbf{e}_1(z) \) and \( \mathbf{e}_2(z) \). We may choose that they equal the local polarization eigenstates at the beginning of our simulations, or we may choose that they equal the local polarization eigenstates.

To study the anisotropy, we start our simulations on the equator of the Poincaré sphere by setting \( S_1(z_0) = (1, 0, 0) \). We have repeated our simulations with \( S_1(z_0) = (0, 1, 0) \), and we find that the results are qualitatively the same. We then repeatedly integrate Eq. \( (1) \) 1000 times with a fixed set of parameters, using different, randomly generated inputs for \( g_r(z) \) in our first model or for \( g_s(z) \) and \( g_p(z) \) in our second model. These 1000 time histories constitute our ensemble that we then use to calculate the variances of \( S_i \), which are the statistical quantities of interest. At large distances the polarization states of the electric field become uniformly distributed on the Poincaré sphere so that \( \langle S_i \rangle \to 0 \) and \( \langle S_i^2 \rangle \to 1/3 \). We define the diffusion length \( d_i \) as the distance at which the variance of \( S_i \), \( \sigma_i^2 = \langle S_i^2 \rangle - \langle S_i \rangle^2 \), rises to \( 1/e \) of its asymptotic value \( 1/3 \), where \( i = 1, 2, 3 \). Quantities that are measured with respect to local polarization eigenaxes are designated by the subscript local, and quantities that are measured with respect to the initial polarization eigenaxes are designated by the subscript fixed. We start in a pure state so that \( \langle \sigma_i^2(z_0) \rangle = 0 \).

In Fig. 1 we plot \( \sigma_i^2 \) versus distance, setting \( h_{fiber} = 0.1 L_B \) in the first model, in which the birefringence strength is fixed. The long-dashed curve gives \( \sigma_1^2_{\text{local}} \), the short-dashed curve gives \( \sigma_2^2_{\text{local}} \), the dotted curves gives \( \sigma_1^2_{\text{fixed}} \), and the dashed–dotted curve gives \( \sigma_2^2_{\text{fixed}} \). The solid curve is the measurement of \( \sigma_3^2 \). We note that the choice of reference axes does not affect \( S_3 \), so that \( \sigma_3^2 \) is the same when measured with respect to both sets of axes. From Fig. 1 it is apparent that the diffusion lengths for the different Stokes parameters are different. We find that in a fiber fluctuation length both \( \sigma_1^2_{\text{local}} \) and \( \sigma_2^2_{\text{local}} \) increase from zero toward \( 1/2 \), the expected variance if there is no azimuthal diffusion. Then both \( \sigma_1^2_{\text{local}} \) and \( \sigma_2^2_{\text{local}} \) approach \( 1/3 \) on the same length scale at which \( \sigma_3^2 \) approaches \( 1/3 \). When measured with respect to the initial eigenaxes, \( \sigma_1^2_{\text{fixed}} \) and \( \sigma_2^2_{\text{fixed}} \) steadily increase toward \( 1/3 \) on a length scale longer than the length scale at which \( \sigma_3^2 \) approaches \( 1/3 \). In other words, the equatorial diffusion length measured with respect to the local eigenaxes is shorter than the azimuthal diffusion length, which in turn is shorter than the equatorial diffusion length measured with respect to the fixed eigenaxes. From a physical standpoint, the electric field cannot follow the rapid changes in the axes of birefringence when \( h_{fiber} < L_B \), and its orientation changes slowly. Thus with respect to the local eigenaxes the electric field will change rapidly, on the length scale \( h_{fiber} \), whereas with respect to the fixed eigenaxes it will change slowly, on a length scale \( L_B^2 / h_{fiber} \).

The same ordering of the diffusion lengths does not hold when the fiber autocorrelation length is comparable with or longer than the beat length. In Fig. 2, we plot \( \sigma_1^2 \) versus distance for \( h_{fiber} = 10 L_B \) in the first model in which the birefringence strength is fixed. The curve types have the same meaning as in Fig. 1. The diffusion length \( d_{1,\text{local}} \) for \( \sigma_1^2_{\text{local}} \) is now the longest. The variance \( \sigma_2^2_{\text{local}} \) and \( \sigma_3^2 \) are almost identical, whereas \( \sigma_1^2_{\text{fixed}} \) and \( \sigma_2^2_{\text{fixed}} \) have small oscillations with a period equal to the beat length. Although these small oscillations appear as sharp steps in Fig. 2, these are adequately resolved numerically because we use 500 points per the shorter of \( h_{fiber} \) and \( L_B \) in our simulations.

In Fig. 3 we plot the diffusion lengths \( d_i \) versus \( h_{fiber} / L_B \) for the first model, in which the birefringence strength is fixed. When measured with respect to the local eigenaxes, the diffusion lengths \( d_{1,\text{local}} \) and \( d_{2,\text{local}} \) are proportional to \( h_{fiber} \), and \( d_{1,\text{local}} \) is larger than \( d_{2,\text{local}} \). With respect to the fixed eigenaxes, we find that the \( d_{1,\text{fixed}} \) are proportional to \( 1 / h_{fiber} \) when \( h_{fiber} \ll L_B \) and proportional to \( h_{fiber} \) when \( h_{fiber} \gg L_B \). When \( h_{fiber} \ll L_B \) we find that \( d_{2,\text{local}} = d_3 \) and when \( h_{fiber} \gg L_B \) we find that \( d_{1,\text{fixed}} = d_{2,\text{fixed}} > d_3 \). In Fig. 4 we plot the diffusion lengths of

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**Fig. 1.** Variances of the Stokes parameters. The \( \sigma_i^2 \) are plotted versus distance for \( h_{fiber} = 0.1 L_B \) in the first model, in which the strength of birefringence is fixed but the orientation varies randomly.

**Fig. 2.** Variances of the Stokes parameters. The \( \sigma_i^2 \) are plotted versus distance for \( h_{fiber} = 10 L_B \) in the first model.
As a simple example of how the anisotropic field diffusion on the Poincaré sphere can affect the nonlinear evolution, we consider the case in which the initial input to the optical fiber is in a single state of linear evolution, we consider the case in which the initial polarization state as $V_0$. The evolution of the strength of the birefringence does not have a significant effect on the diffusion lengths $d_i$.

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Fig. 3. Diffusions length $d_i$ versus $h_{fiber}/L_B$ for the first model.

Fig. 4. Diffusions length $d_i$ versus $h_{fiber}/L_B$ for the first model, in which both the strength and the orientation of the birefringence vary randomly.

Fig. 5. Diffusions length $d_i$ versus $h_{fiber}/L_B$ for the second model, in which both the birefringence strength and orientation vary randomly.

The results are qualitatively similar to those of the first model, shown in Fig. 3, so that including the variation of the strength of the birefringence does not have a significant effect on the diffusion lengths $d_i$.

In conclusion, using two physically reasonable models, we show that the diffusion length of the Stokes parameters is a function of both $h_{fiber}$ and $L_B$. We have demonstrated that the diffusion is anisotropic in general. This anisotropy can have important consequences for the nonlinear evolution of light in optical fibers and for current practice in simulations that randomize the electric field on the Poincaré sphere at a fixed interval.

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References
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